GRADED LIMITS OF SIMPLE TENSOR PRODUCT OF KIRILLOV-RESHETIKHIN MODULES FOR $U_q(\widetilde{\mathfrak{sl}}_{n+1})$

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Abstract. We study graded limits of simple $U_q(\widetilde{\mathfrak{sl}}_{n+1})$-modules which are isomorphic to tensor products of Kirillov-Reshetikhin modules associated to a fixed fundamental weight. We prove that every such module admits a graded limit which is isomorphic to the fusion product of the graded limits of its tensor factors. Moreover, using recent results of Naoi, we exhibit a set of defining relations for these graded limits.

Introduction

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra, $\widetilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ the corresponding loop algebra, and $U_q(\mathfrak{g})$, $U_q(\widetilde{\mathfrak{g}})$ their Drinfeld–Jimbo quantum groups over $\mathbb{C}(q)$, where q is an indeterminate. Despite the classification of the simple $U_q(\widetilde{\mathfrak{g}})$-modules being known since [9] and the great development of the theory ever since, several other basic questions about the structure of these representations remain essentially unanswered. One of the methods which have been used to study the structure of simple $U_q(\widetilde{\mathfrak{g}})$-modules is to understand the classical limit of these representations, i.e., their specialization at 1 of the quantum parameter q, and regard it as a representation for the current algebra $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$. This approach was first considered in [14], where the authors proved necessary and sufficient condition of the existence of the limit. The notion of the graded limit of a $U_q(\widetilde{\mathfrak{g}})$-module was then further developed in [3, 8].

In [2], Chari introduced an important class of finite–dimensional $U_q(\widetilde{\mathfrak{g}})$-modules called minimal affinizations. The Kirillov–Reshetikhin modules are the minimal affinizations of simple modules whose highest weights are multiples of a fundamental weight. In [3, 8], the authors proved that the Kirillov–Reshetikhin modules admit graded limits, described a set of defining relations for them and computed their graded characters. The graded limits of general minimal affinizations were first studied in [26] where it was conjectured a set of defining relations for them. Using the theory of Demazure modules, the conjecture was established in [30, 31] for $\mathfrak{g}$ of classical type and in [24] for type G_2. It was also partially established for type E_6 in [27].

It is not hard to see that if V and W are simple $U_q(\widetilde{\mathfrak{g}})$-modules, then the classical limit of $V \otimes W$ is not isomorphic to the tensor product of the classical limits of V and W. In [18], Feigin and Loktev introduced the notion of the fusion product of graded representations of the current algebra. It was proved in [6, 19, 28] that a local Weyl module for $\mathfrak{g}[t]$ is isomorphic to

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a fusion product of fundamental local Weyl modules. On the other hand, it was known that the quantum local Weyl modules are isomorphic to tensor products of fundamental local Weyl modules (see [7, Section 7.4]). Therefore, it follows that the graded limit of a tensor product of quantum local Weyl modules is isomorphic to the fusion product of the graded limits of the corresponding factors. This result motivates the following question: Is it true that if a module V for $U_q(\widetilde{\mathfrak{g}})$ is isomorphic to a tensor product $V_1 \otimes V_2$ and all three modules admit graded limits, then the graded limit of V is isomorphic to the fusion product of the graded limits of V_1 and V_2? There are a few positive answers for this question in the case that V belongs to certain subclasses of simple modules. See for instance [16, Section 4 and 5] where V is isomorphic to particular tensor products of Kirillov–Reshetikhin modules and [1] for modules whose prime factors belong to a special subcategory of modules considered by Hernandez and Leclerc in [20, 21].

In this paper we prove that the answer to the above question is positive in the case that $\mathfrak{g}$ is of type A and V is a tensor product of Kirillov–Reshetikhin modules associated to an arbitrary fixed fundamental weight (Theorem 3.1). Moreover, as consequence of the results of [32], this $\mathfrak{g}[t]$-module is isomorphic to a generalized Demazure module and a $\mathfrak{g}[t]$-module given by generators and relations (Corollary 3.3).

The paper is organized as follows. In Section 1 we give some background information about Lie algebras and their representations. In Section 2 we briefly recall some relevant facts about the finite-dimensional representations of quantum loop algebras. In Section 3 we state and prove our main results.

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1. Lie algebras

Throughout the paper, let $\mathbb{C}, \mathbb{Z}, \mathbb{Z}_{\geq m}$ denote the sets of complex numbers, integers and integers bigger than or equal to m, respectively. Given a ring $\mathbf{R}$, the underlying multiplicative group of units is denoted by $\mathbf{R}^\times$. Given any complex Lie algebra $\mathfrak{a}$ we let $U(\mathfrak{a})$ be the universal enveloping algebra of $\mathfrak{a}$.

1.1. **Basics and notation.** Let $\mathfrak{g}$ be a complex simple Lie algebra of rank n and $\mathfrak{h}$ a Cartan subalgebra. We identify $\mathfrak{h}$ and $\mathfrak{h}^*$ by means of the invariant inner product $(\cdot,\cdot)$ on $\mathfrak{g}$ normalized such that the square length of the maximal root equals 2. Let $I = \{1, \dots, n\}$ and R^+ be the set of positive roots of $\mathfrak{g}$. We denote by $\{\alpha_i\}_{i\in I}$ and $\{\omega_i\}_{i\in I}$, the sets of simple roots and fundamental weights, respectively, while Q, P, Q^+, P^+ the root and weight lattices with corresponding positive cones.

We fix a Chevalley basis of $\mathfrak{g}$ consisting of $x_\alpha^\pm \in \mathfrak{g}_{\pm\alpha}$, for each $\alpha \in R^+$, and $h_i \in \mathfrak{h}$, $i \in I$. We also define $h_\alpha \in \mathfrak{h}$, $\alpha \in R^+$, by $h_\alpha = [x_\alpha^+, x_\alpha^-]$. We often simplify notation and write $x_i^\pm$ in place of $x_{\alpha_i}^\pm$, $i \in I$. Let $r^\vee$ be the maximal number of edges connecting two vertices of the Dynkin diagram of $\mathfrak{g}$ and let also

$$d_\alpha = \frac{r^\vee}{2}(\alpha,\alpha), \quad \check{d}_\alpha = \frac{r^\vee}{d_\alpha}, \quad d_i = d_{\alpha_i}, \quad \alpha \in R^+, i \in I.$$

Recall that, if $C = (c_{ij})_{i,j\in I}$ is the Cartan matrix of $\mathfrak{g}$, i.e., $c_{ij} = \alpha_j(h_i)$, then $d_ic_{ij} = d_jc_{ji}$.

Define the loop algebra of $\mathfrak{g}$ by $\widetilde{\mathfrak{g}} = \mathfrak{g}\otimes_{\mathbb{C}}\mathbb{C}[t,t^{-1}]$ with bracket given by $[x\otimes t^r, y\otimes t^s] = [x,y]\otimes t^{r+s}$. We identify $\mathfrak{g}$ with the subalgebra $\mathfrak{g}\otimes 1$ of $\widetilde{\mathfrak{g}}$, hence, we will continue denoting its elements by x instead of $x\otimes 1$. The subalgebra $\mathfrak{g}[t] = \mathfrak{g}\otimes\mathbb{C}[t]$ of $\widetilde{\mathfrak{g}}$ is the current algebra associated to $\mathfrak{g}$.

If $\mathfrak{a}$ is a subalgebra of $\mathfrak{g}$, let $\mathfrak{a}[t] = \mathfrak{a}\otimes\mathbb{C}[t]$ and its ideal $\mathfrak{a}[t]_+ = \mathfrak{a}\otimes t\mathbb{C}[t]$. The degree grading on $\mathbb{C}[t]$ defines a natural $\mathbb{Z}_{\geq 0}$-grading on $\mathfrak{a}[t]$ and thus, also on $U(\mathfrak{a}[t])$. An element of the form $(a_1\otimes t^{r_1})\cdots(a_s\otimes t^{r_s})$ has grade $r_1+\cdots+r_s$ and we denote by $U(\mathfrak{a}[t])[r]$ the subspace of grade r.

The affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ is the Lie algebra with underlying vector space $\widetilde{\mathfrak{g}}\oplus\mathbb{C}c\oplus\mathbb{C}d$ equipped with the Lie bracket given by

$$[x\otimes t^r, y\otimes t^s] = [x,y]\otimes t^{r+s} + r\delta_{r,-s}(x,y)c, \quad [c,\widehat{\mathfrak{g}}] = 0 \quad \text{and} \quad [d, x\otimes t^r] = rx\otimes t^r,$$

for any $x,y\in\mathfrak{g}$, $r,s\in\mathbb{Z}$. A Cartan subalgebra $\widehat{\mathfrak{h}}$ and a Borel subalgebra $\widehat{\mathfrak{b}}$ are defined as follows:

$$\widehat{\mathfrak{h}} = \mathfrak{h}\oplus\mathbb{C}c\oplus\mathbb{C}d, \qquad \widehat{\mathfrak{b}} = \widehat{\mathfrak{h}}\oplus\mathfrak{n}^+\oplus\mathfrak{g}\otimes t\mathbb{C}[t].$$

Set $\widehat{\mathfrak{n}}^+ = \mathfrak{n}^+\oplus\mathfrak{g}\otimes t\mathbb{C}[t]$.

We often consider $\mathfrak{h}^*$ as a subspace of $\widehat{\mathfrak{h}}^*$ by setting $\lambda(c) = \lambda(d) = 0$ for $\lambda\in\mathfrak{h}^*$. The root system and positive root associated to the triangular decomposition $\widehat{\mathfrak{g}} = \widehat{\mathfrak{n}}^-\oplus\widehat{\mathfrak{h}}\oplus\widehat{\mathfrak{n}}^+$ will be denoted by $\widehat{R}$, $\widehat{R}^+$, respectively. Let $\theta\in R^+$ be the highest root, $\delta\in\widehat{\mathfrak{h}}^*$ be such that $\delta(d) = 1$, $\delta(c) = \delta(h) = 0$, $h\in\mathfrak{h}$, and $\alpha_0 = -\theta+\delta$. Then, if we set $\widehat{I} = I\sqcup\{0\}$, we have that $\widehat{\Delta} = \{\alpha_i\}_{i\in\widehat{I}}$ is the set of simple roots of $\widehat{\mathfrak{g}}$, and

$$\widehat{R}^+ = (R+\mathbb{Z}_{\geq 1}\delta)\cup R^+\cup\mathbb{Z}_{\geq 1}\delta.$$

The elements $x^\pm_\alpha\otimes t^r$, $x^\pm_i\otimes t^r$, and $h_i\otimes t^r$ will be denoted by $x^\pm_{\alpha,r}$, $x^\pm_{i,r}$, and $h_{i,r}$, respectively. Set also, $x_0^\pm = x^\mp_{\theta,\pm 1}$. Then

$$h_0 := [x_0^+, x_0^-] = c - h_\theta.$$

Let $\widehat{Q} = \oplus_{i\in\widehat{I}}\mathbb{Z}\alpha_i$ and $\widehat{Q}^+ = \oplus_{i\in\widehat{I}}\mathbb{Z}_{\geq 0}\alpha_i$. Let also $\Lambda_0\in\widehat{\mathfrak{h}}^*$ be the unique element satisfying $\Lambda_0(c) = 1$ and $\Lambda_0(\mathfrak{h}) = \Lambda_0(d) = 0$. Then $\widehat{\mathfrak{h}}^* = \mathfrak{h}\oplus\mathbb{C}\delta\oplus\mathbb{C}\Lambda_0$. Define $\Lambda_i\in\widehat{\mathfrak{h}}^*$, $i\in I$, by the requirement $\Lambda_i(d) = 0$, $\Lambda_i(h_i) = \delta_{i,j}$, $j\in\widehat{I}$, and note that $\Lambda_i = \omega_i + \omega_i(h_\theta)\Lambda_0$, for $i\in I$. Let $\widehat{P} = \oplus_{i=0}^n\mathbb{Z}\Lambda_i\oplus\mathbb{Z}\delta$ and $\widehat{P}^+ = \oplus_{i=0}^n\mathbb{Z}_{\geq 0}\Lambda_i\oplus\mathbb{Z}\delta$. Equip $\widehat{\mathfrak{h}}^*$ with the partial order $\lambda\leq\mu$ if and only if $\mu-\lambda\in\widehat{Q}^+$. Let $\widehat{\mathcal{W}}$ denote the affine Weyl group, which is generated by the simple reflections s_i, $i\in\widehat{I}$, where

$$s_i(\mu) = \mu - \mu(h_i)\alpha_i, \quad \mu\in\widehat{\mathfrak{h}}^*.$$

The length of $w\in\widehat{\mathcal{W}}$ will be denoted by $\ell(w)$. Recall that the subgroup of $\widehat{\mathcal{W}}$ generated by $s_i, i\in I$, is the Weyl group $\mathcal{W}$ of $\mathfrak{g}$ and we denote its longest element by w_0. Let $L = \oplus_{i\in I}\check{d}_{\alpha_i}\omega_i$ be the co-weight lattice and $M = \oplus_{i\in I}\check{d}_{\alpha_i}\alpha_i$ the co–root lattice. Given $\mu\in\mathfrak{h}^*$, we define $t_\alpha\in GL(\widehat{\mathfrak{h}}^*)$ by

$$t_\mu(\lambda) = \lambda - (\lambda,\mu)\delta, \quad \lambda\in\mathfrak{h}^*\oplus\mathbb{C}\delta, \quad t_\mu(\Lambda_0) = \Lambda_0 + \mu - \frac{1}{2}(\mu,\mu)\delta. \tag{1.1}$$

Defining $T_M = \{t_\mu \in GL(\widehat{\mathfrak{h}}^*) | \mu \in M\}$ we have $\widehat{\mathcal{W}} = \mathcal{W} \ltimes T_M$. The extended affine Weyl group $\widetilde{\mathcal{W}}$ is the semi–direct product $\mathcal{W} \ltimes T_L$, where $T_L = \{t_\mu \in GL(\widehat{\mathfrak{h}}^*) | \mu \in L\}$. We also have $\widetilde{\mathcal{W}} = \widehat{\mathcal{W}} \ltimes \mathcal{T}$, where $\mathcal{T}$ is the group of diagram automorphisms of $\widehat{\mathfrak{g}}$. The length function ℓ is extended to $\widetilde{\mathcal{W}}$ by setting $\ell(w\tau) = \ell(w)$, for all $w \in \widehat{\mathcal{W}}$ and $\tau \in \mathcal{T}$. The following lemma was proved in [15].

Lemma 1.1. *Given $\lambda, \mu \in P^+$ and $w \in \mathcal{W}$, we have*

$$\ell(t_{-\lambda} t_{-\mu} w) = \ell(t_{-\lambda}) + \ell(t_{-\mu} w).$$

1.2. **Graded $\mathfrak{g}[t]$-modules.** A graded representation of $\mathfrak{g}[t]$ is a $\mathbb{Z}$-graded vector space which admits a compatible Lie algebra action of $\mathfrak{g}[t]$, i.e.,

$$V = \bigoplus_{r \in \mathbb{Z}} V[r], \qquad (\mathfrak{g} \otimes t^r) V[s] \subseteq V[s+r], \quad s \in \mathbb{Z}, \quad r \in \mathbb{Z}_{\geq 0}.$$

A graded morphism of $\mathfrak{g}[t]$-modules is a degree zero morphism of $\mathfrak{g}[t]$-graded modules. For $r \in \mathbb{Z}$, we let $\tau_r V$ be the r-th graded shift of V. Given $a \in \mathbb{C}$, let $\operatorname{ev}_a : \mathfrak{g}[t] \to \mathfrak{g}$ be the evaluation map $x \otimes f(t) \mapsto f(a)x$. Therefore, if W is a $\mathfrak{g}$-module we can define a $\mathfrak{g}[t]$-module structure on W by taking the pull–back by ev_a. This $\mathfrak{g}[t]$-module is denoted by $\operatorname{ev}_a W$ and it is clearly irreducible if and only if W is an irreducible $\mathfrak{g}$-module. Moreover, $\operatorname{ev}_0 W$ is a graded $\mathfrak{g}[t]$-module such that

$$(\operatorname{ev}_0 W)[0] = W \quad \text{and} \quad U(\mathfrak{g} \otimes t\mathbb{C}[t])(\operatorname{ev}_0 W) = 0.$$

1.3. **Demazure modules.** Recall that a weight module for $\widehat{\mathfrak{h}}$ is one where $\widehat{\mathfrak{h}}$ acts diagonally. For $\Lambda \in \widehat{P}^+$, let $\widehat{V}(\Lambda)$ be the irreducible highest weight integrable $\widehat{\mathfrak{g}}$-module generated by an element v_Λ with defining relations

$$\widehat{\mathfrak{n}}^+ v_\Lambda = 0, \quad h_i v_\Lambda = \Lambda(h_i) v_\Lambda, \quad (x^-_{\alpha_i})^{\Lambda(h_i)+1} v_\Lambda = 0, \quad i \in \widehat{I}.$$

Then

$$\widehat{V}(\Lambda)_\mu \neq \{0\} \quad \text{only if} \quad \mu \in \Lambda - \widehat{Q}^+.$$

The following proposition is well–known (see [22, Chapters 10,11] for instance).

Proposition 1.2. *(i) Let $\Lambda \in \widehat{P}^+$. Then*

$$\dim V(\Lambda)_{w\Lambda} = 1, \quad \text{for all} \quad w \in \widehat{W}.$$

(ii) Given $\Lambda', \Lambda'' \in \widehat{P}^+$, let $\Lambda = \Lambda' + \Lambda''$. Then

$$\dim \operatorname{Hom}_{\widehat{\mathfrak{g}}} \left(V(\Lambda), V(\Lambda') \otimes V(\Lambda'') \right) = \begin{cases} 1, & \Lambda = \Lambda' + \Lambda'', \\ 0, & \Lambda \notin \Lambda' + \Lambda'' - \widehat{Q}^+. \end{cases}$$

Moreover, for all $w \in \widehat{W}$, we have

$$\left(\widehat{V}(\Lambda') \otimes \widehat{V}(\Lambda'') \right)_{w\Lambda} = \widehat{V}(\Lambda)_{w\Lambda},$$

where we have identified $\widehat{V}(\Lambda)$ with its image in $\widehat{V}(\Lambda') \otimes \widehat{V}(\Lambda'')$.

Given $\Lambda \in \widehat{P}^+$ and $w\tau \in \widetilde{\mathcal{W}}$, with $w \in \widehat{\mathcal{W}}$ and $\tau \in \mathcal{T}$, the subspace $\widehat{V}(\tau\Lambda)_{w\tau\Lambda}$ is one-dimensional and we fix a non-zero vector $v_{w\tau\Lambda}$ of this weight space. The Demazure module $D(w\tau\Lambda)$ is the $\widehat{\mathfrak{b}}$-module of $\widehat{V}(\tau\Lambda)$ defined by

$$D(w\tau\Lambda) = U(\widehat{\mathfrak{b}})v_{w\tau\Lambda}.$$

In this paper we consider the following generalization of Demazure modules introduced in [30]. Given $m \in \mathbb{Z}_{\geq 1}$ and pairs $(w_r, \Lambda^r) \in \widetilde{\mathcal{W}} \times \widehat{P}^+$, $1 \leq r \leq m$, set

$$D(w_1\Lambda^1, \dots, w_m\Lambda^m) = U(\widehat{\mathfrak{b}})(v_{w_1\Lambda^1} \otimes \cdots \otimes v_{w_m\Lambda^m}) \subseteq D(w_1\Lambda^1) \otimes \cdots \otimes D(w_m\Lambda^m).$$

Our primary focus in this paper are Demazure modules and its generalizations such that $w_r\Lambda^r(h_i) \leq 0$, for all $i \in I$, $1 \leq r \leq m$. In this case we have $\mathfrak{n}^- v_{w_r\Lambda^r} = 0$, $1 \leq r \leq m$, and then $D(w_1\Lambda^1, \dots, w_m\Lambda^m)$ is a module for the parabolic subalgebra $\widehat{\mathfrak{b}} \oplus \mathfrak{n}^-$, i.e.,

$$D(w_1\Lambda^1, \dots, w_m\Lambda^m) = U(\mathfrak{g}[t])(v_{w_0w_1\Lambda^1} \otimes \cdots \otimes v_{w_0w_m\Lambda^m}), \tag{1.2}$$

c.f. [34, Proposition 16]. Given $(\ell, \lambda) \in \mathbb{Z}_{\geq 1} \times P^+$, there exists unique $\Lambda \in \widehat{P}^+$ and $w \in \widetilde{\mathcal{W}}$ such that

$$w\Lambda = w_0\lambda + \ell\Lambda_0$$

and we shall denote the $\mathfrak{g}[t]$-module $D(w\Lambda)$ by $D(\ell, \lambda)$. In [16], it is given a finite presentation for the $\mathfrak{g}[t]$-modules $D(\ell, \lambda)$ which we recall now for the simply laced case (see [16, Theorem 2] for complete generality).

Proposition 1.3. *Assume that $\mathfrak{g}$ is simply laced and let $(\ell, \lambda) \in \mathbb{Z}_{\geq 1} \times P^+$. The $\mathfrak{g}[t]$-module $D(\ell, \lambda)$ is generated by an element $v_{\ell,\lambda}$ satisfying the following defining relation:*

$$(x_i^+ \otimes 1)v_{\ell,\lambda} = 0, \quad (h_i \otimes t^r)v_{\ell,\lambda} = \lambda(h_i)\delta_{r,0}v_{\ell,\lambda}, \quad (x_i^-)^{\lambda(h_i)+1}v_{\ell,\lambda} = 0, \tag{1.3}$$
$$(x_\alpha^- \otimes t^{s_\alpha})v_{\ell,\lambda} = 0, \quad \alpha \in R^+, \tag{1.4}$$
$$(x_\alpha^- \otimes t^{s_\alpha - 1})^{m_\alpha + 1}v_{\ell,\lambda} = 0, \quad \alpha \in R^+, \tag{1.5}$$

where $\lambda(h_\alpha) = (s_\alpha - 1)\ell + m_\alpha$, with $0 < m_\alpha \leq \ell$. Moreover, if $m_\alpha = \ell$, (1.5) *is a consequence of* (1.3) *and* (1.4)*. In the particular case when $\ell = 1$, the relations* (1.4) *and* (1.5) *follow from* (1.3)*.*

We declare the grade of $v_{\ell,\lambda}$ to be zero and, since the defining relations of $D(\ell, \lambda)$ are graded, it follows that $D(\ell, \lambda)$ is a graded $\mathfrak{g}[t]$-module. The following is a consequence of Proposition 1.2 (ii).

Lemma 1.4. *Let $\lambda \in P^+$ and $\ell \in \mathbb{Z}_{\geq 1}$. Then*

$$D(\ell, \ell\lambda) \cong_{\mathfrak{g}[t]} U(\mathfrak{g}[t])v_{1,\lambda}^{\otimes \ell} \subseteq D(1, \lambda)^{\otimes \ell}.$$

□

1.4. **Fusion product.** We recall the notion of fusion product of finite dimensional cyclic graded $\mathfrak{g}[t]$–modules introduced in [18]. Let V be a finite–dimensional cyclic $\mathfrak{g}[t]$–module generated by an element v. We define a filtration F^rV, $r \in \mathbb{Z}_{\geq 0}$, on V by

$$F^rV = \left(\bigoplus_{0 \leq s \leq r} U(\mathfrak{g}[t])[s] \right) \cdot v.$$

The associated graded vector space $\mathrm{gr}V$ acquires a graded $\mathfrak{g}[t]$–module structure in a natural way and is generated by the image of v in $\mathrm{gr}V$.

Let $p \in \mathbb{Z}_{\geq 1}$. Let $\lambda_1, \ldots, \lambda_p$ be a sequence of elements of P^+ and $z_1, \ldots, z_p$ pairwise distinct complex numbers. Then

$$\mathbf{V}(\mathbf{z}) := \mathrm{ev}_{z_1} V(\lambda_1) \otimes \cdots \otimes \mathrm{ev}_{z_p} V(\lambda_p),$$

is a finite dimensional cyclic $\mathfrak{g}[t]$-module, where $V(\lambda_s)$ is the finite–dimensional irreducible $\mathfrak{g}$-module of highest weight λ_s, $1 \leq s \leq p$. Then, the $\mathfrak{g}[t]$-module $\mathrm{gr}\mathbf{V}(\mathbf{z})$ is called the fusion product of $V(\lambda_1), \ldots, V(\lambda_p)$ and denote by

$$V(\lambda_1) * \cdots * V(\lambda_p).$$

Clearly the definition of the fusion product depends on the parameters z_s, $1 \leq s \leq p$. However it is conjectured in [18] and (proved in certain cases by various people, [6], [17], [18] [19], [23] for instance) that the fusion product is independent of the choice of the complex numbers, hence we suppress this dependence in our notation. Note that, by definition we have

$$V(\lambda_1) * \cdots * V(\lambda_p) \cong_{\mathfrak{g}} V(\lambda_1) \otimes \cdots \otimes V(\lambda_p). \tag{1.6}$$

2. Quantum algebras and graded limits

2.1. **Basics and notation.** We give a brief reminder on quantum loops algebras and their finite-dimensional representations. We refer the reader to [11] for the basic definitions.

Let $\mathbb{C}(q)$ be the field of rational functions in an indeterminate q and $\mathbb{A} = \mathbb{Z}[q, q^{-1}]$. Let $U_q(\mathfrak{g})$ and $U_q(\widetilde{\mathfrak{g}})$ be the quantized enveloping algebras over $\mathbb{C}(q)$ associated to $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$, respectively. The algebra $U_q(\mathfrak{g})$ is isomorphic to a subalgebra of $U_q(\widetilde{\mathfrak{g}})$. Let $U_{\mathbb{A}}(\mathfrak{g})$ and $U_{\mathbb{A}}(\widetilde{\mathfrak{g}})$ be the $\mathbb{A}$-form of $U_q(\mathfrak{g})$ and $U_q(\widetilde{\mathfrak{g}})$ defined in [25]. These are free subalgebras such that

$$U_q(\mathfrak{g}) \cong U_{\mathbb{A}}(\mathfrak{g}) \otimes_{\mathbb{A}} \mathbb{C}(q) \quad U_q(\widetilde{\mathfrak{g}}) \cong U_{\mathbb{A}}(\widetilde{\mathfrak{g}}) \otimes_{\mathbb{A}} \mathbb{C}(q).$$

Regarding $\mathbb{C}$ to be the $\mathbb{A}$-module by letting q act as 1, the algebras $U_{\mathbb{A}}(\mathfrak{g}) \otimes_{\mathbb{A}} \mathbb{C}$ and $U_{\mathbb{A}}(\widetilde{\mathfrak{g}}) \otimes_{\mathbb{A}} \mathbb{C}$ over $\mathbb{C}$ have $U(\mathfrak{g})$ and $U(\widetilde{\mathfrak{g}})$ as canonical quotients. We also recall that $U_q(\widetilde{\mathfrak{g}})$ is a Hopf algebra and that $U_{\mathbb{A}}(\widetilde{\mathfrak{g}})$, $U_q(\mathfrak{g})$ and $U_{\mathbb{A}}(\mathfrak{g})$ are Hopf subalgebras.

It is well known that the isomorphism classes of irreducible finite-dimensional representations of $U_q(\mathfrak{g})$ are indexed by elements of P^+. Given $\lambda \in P^+$, we denote by $V_q(\lambda)$ an element of the corresponding isomorphism class. Moreover, the category of finite-dimensional $U_q(\mathfrak{g})$-modules is semisimple.

Let $\mathcal{P}_q^+$ be the multiplicative monoid of n-tuples of polynomials $\boldsymbol{\pi} = (\pi_1(u), \ldots, \pi_n(u))$, $\pi_i(u) \in \mathbb{C}(q)[u]$, for an indeterminate u, such that $\pi_i(0) = 1$ for all $i \in I$. We shall only be interested in the submonoid $\mathcal{P}^+$ of elements $\boldsymbol{\pi} \in \mathcal{P}_q^+$ such that $\pi_i(u)$ splits into linear factors in $\mathbb{C}(q)$, for all $i \in I$.

Given $a \in \mathbb{C}(q)^\times$ and $i \in I$, let $\boldsymbol{\omega}_{i,a} \in \mathcal{P}^+$ be the fundamental ℓ-weights, defined by

$$(\boldsymbol{\omega}_{i,a})_j(u) = 1 - \delta_{i,j} au.$$

Observe that $\mathcal{P}^+$ is the free abelian monoid generated by $\{\boldsymbol{\omega}_{i,a} : i \in I, a \in \mathbb{C}(q)^\times\}$, and denote by $\mathcal{P}$ the corresponding free abelian group. Let also $\mathcal{P}^+_{\mathbb{Z}}$ be the submonoid of $\mathcal{P}^+$ of elements $\boldsymbol{\pi} \in \mathcal{P}^+$ where $\pi_i(u)$ has its roots in $q^{\mathbb{Z}}$, for all $i \in I$.

Consider the group homomorphism (weight map) $\mathrm{wt} : \mathcal{P} \to P$ by setting $\mathrm{wt}(\boldsymbol{\omega}_{i,a}) = \omega_i$.

It was proved in [9, 12, 13] that the isomorphism classes of irreducible finite-dimensional representations of $U_q(\widetilde{\mathfrak{g}})$ is indexed by $\mathcal{P}^+_q$. Given $\boldsymbol{\pi} \in \mathcal{P}^+_q$, we let $L_q(\boldsymbol{\pi})$ be an irreducible representation in the corresponding isomorphism class. The module $L_q(\boldsymbol{\pi})$ is said to be an affinization of $V_q(\lambda)$ if $\mathrm{wt}(\boldsymbol{\pi}) = \lambda$. Two simple $U_q(\widetilde{\mathfrak{g}})$-modules are said to be equivalent if they are isomorphic as $U_q(\mathfrak{g})$-modules.

It will be convenient to introduce the following notation. Given $i \in I, a \in \mathbb{C}^\times, m \in \mathbb{Z}_{\geq 1}$, define

$$\boldsymbol{\omega}_{i,a,m} = \prod_{j=0}^{m-1} \boldsymbol{\omega}_{i,aq^{d_i(m-1-2j)}}.$$

The modules $L_q(\boldsymbol{\omega}_{i,a,m})$ are called Kirillov-Reshetikhin modules. Given $\boldsymbol{\pi} \in \mathcal{P}^+$, there exist unique $m_i \in \mathbb{Z}_{\geq 0}$, $a_{ik} \in \mathbb{C}(q)^\times$ and $r_{ik} \in \mathbb{Z}_{\geq 1}$ such that

$$\boldsymbol{\pi} = \prod_{i \in I} \prod_{k=1}^{m_i} \boldsymbol{\omega}_{i,a_{ik},r_{ik}}$$

with

$$\frac{a_{ij}}{a_{il}} \neq q^{\pm d_i(r_{ij}+r_{il}-2p)} \quad \text{and} \quad \sum_{k=1}^{m_i} r_{ik} = \mathrm{wt}(\boldsymbol{\omega})(h_i)$$

for all $i \in I$, $j \neq l$ and $0 \leq p < \min\{r_{ij}, r_{il}\}$. This decomposition is called q-factorization of $\boldsymbol{\pi}$.

In the next theorem we collect important results of $U_q(\widetilde{\mathfrak{sl}}_{n+1})$-modules. The first item is [10, Theorem 3.5], and the second item is the dual of [4, Theorem 6.1, Corollary 6.2], for our case of interest.

Theorem 2.1. *Assume* $\mathfrak{g} = \mathfrak{sl}_{n+1}$.

(i) For all $i \in I$, $a \in \mathbb{C}(q)^\times$ *and* $m \in \mathbb{Z}_{\geq 0}$ *we have* $L_q(\boldsymbol{\omega}_{i,a,m}) \cong_{U_q(\mathfrak{g})} V_q(m\omega_i)$.

(ii) Let $m \in \mathbb{Z}_{\geq 1}$, $i_j \in I$, $a_j \in \mathbb{C}(q)^\times$, $n_j \in \mathbb{Z}_{\geq 1}$, $1 \leq j \leq m$, *be such that*

$$r > s \Longrightarrow \frac{a_s}{a_r} \neq q^{n_s+n_r+2-2p+2k-i_r-i_s}, \tag{2.1}$$

for all $1 \leq p \leq \min\{n_s, n_r\}$ *and* $\min\{i_r, i_s\} < k+1 \leq \min\{i_r+i_s, n+1\}$. *Then* $L_q(\prod_{j=1}^m \boldsymbol{\omega}_{i_j,a_j,n_j})$ *is the unique irreducible submodule of*

$$L_q(\boldsymbol{\omega}_{i_1,a_1,n_1}) \otimes \ldots \otimes L_q(\boldsymbol{\omega}_{i_m,a_m,n_m}). \tag{2.2}$$

Moreover, if (2.1) *holds for all* $1 \leq r, s \leq m$, *then the module described in* (2.2) *is irreducible.*

In particular, if $\boldsymbol{\pi} = \prod_{j=1}^{m} \boldsymbol{\omega}_{i,a_j,n_j}$ *is its q-factorization, then*

$$L_q(\boldsymbol{\pi}) \cong_{U_q(\widetilde{\mathfrak{g}})} L_q(\boldsymbol{\omega}_{i,a_1,n_1}) \otimes \cdots \otimes L_q(\boldsymbol{\omega}_{i,a_m,n_m}).$$

□

2.2. **The modules $L(\boldsymbol{\pi})$, $\boldsymbol{\pi} \in \mathcal{P}_{\mathbb{Z}}^+$.** In this section we assume that $\mathfrak{g}$ is of classical type. We recall the definition of the $\mathfrak{g}[t]$–modules $L(\boldsymbol{\pi}), \boldsymbol{\pi} \in \mathcal{P}_{\mathbb{Z}}^+$. We refer the reader to [1, Section 2.1] and references therein for a detailed exposition.

It was shown in [14, Section 4] that, given $\boldsymbol{\pi} \in \mathcal{P}_{\mathbb{Z}}^+$, the module $L_q(\boldsymbol{\pi})$ admits an $\mathbb{A}$-form $L_{\mathbb{A}}(\boldsymbol{\pi})$ and there is an action of $\widetilde{\mathfrak{g}}$ on $\overline{L_q(\boldsymbol{\pi})} := L_{\mathbb{A}}(\boldsymbol{\pi}) \otimes_{\mathbb{A}} \mathbb{C}$. Moreover, as $\widetilde{\mathfrak{g}}$-module, $\overline{L_q(\boldsymbol{\pi})}$ is generated by a vector $v_{\overline{\boldsymbol{\pi}}}$ which satisfies the relations:

$$x_{i,s}^+ v_{\overline{\boldsymbol{\pi}}} = 0, \quad h_{i,r} v_{\overline{\boldsymbol{\pi}}} = \mathrm{wt}(\boldsymbol{\pi})(h_i) v_{\overline{\boldsymbol{\pi}}}, \quad (x_{i,0}^-)^{\mathrm{wt}(\boldsymbol{\pi})(h_i)+1} v_{\overline{\boldsymbol{\pi}}} = 0.$$

By restricting the action of $\widetilde{\mathfrak{g}}$ to the subalgebra $\mathfrak{g}[t]$ one can regard $\overline{L_q(\boldsymbol{\pi})}$ as a module of $\mathfrak{g}[t]$ generated by $v_{\overline{\boldsymbol{\pi}}}$. The $\mathfrak{g}[t]$-module $L(\boldsymbol{\pi})$ is then defined by the pullback of $\overline{L_q(\boldsymbol{\pi})}$ by the $\mathfrak{g}[t]$-automorphism $x \otimes f(t) \to x \otimes f(t-1)$.

In sum, the following is a consequence of [14, Section 4] and the main result of [6].

Theorem 2.2. *For* $\boldsymbol{\pi} \in \mathcal{P}_{\mathbb{Z}}^+$, *the* $\mathfrak{g}[t]$-*module* $L(\boldsymbol{\pi})$ *is generating by an element* $v_{\overline{\boldsymbol{\pi}}}$ *satisfying*

$$(x_i^+ \otimes \mathbb{C}[t]) v_{\overline{\boldsymbol{\pi}}} = 0, \qquad (h_i \otimes t^r) v_{\overline{\boldsymbol{\pi}}} = \delta_{r,0} \mathrm{wt}(\boldsymbol{\pi})(h_i) v_{\overline{\boldsymbol{\pi}}}, \qquad (x_i^- \otimes 1)^{\mathrm{wt}(\boldsymbol{\pi})(h_i)+1} v_{\overline{\boldsymbol{\pi}}} = 0. \tag{2.3}$$

Moreover,

(i) $\dim L_q(\boldsymbol{\pi}) = \dim L(\boldsymbol{\pi})$,
(ii) if $L_q(\boldsymbol{\pi}) \cong V_q(\mathrm{wt}(\boldsymbol{\pi}))$, *then* $L(\boldsymbol{\pi}) \cong_{\mathfrak{g}[t]} \mathrm{ev}_0 V(\mathrm{wt}(\boldsymbol{\pi}))$, *and*
(iii) if

$$L_q(\boldsymbol{\pi}) \cong_{U_q(\widetilde{\mathfrak{g}})} L_q(\boldsymbol{\omega}_{i_1,a_1}) \otimes \cdots \otimes L_q(\boldsymbol{\omega}_{i_p,a_p}),$$

for some $p \in \mathbb{Z}_{\geq 1}$ *and* $(i_j, a_j) \in I \times q^{\mathbb{Z}}$, $1 \leq j \leq p$, *then the relations* (2.3) *are defining relations of* $L(\boldsymbol{\pi})$.

□

The next result will be very useful in the proof of our main result (see [26, Lemma 2.20 and proof of Proposition 3.21]).

Lemma 2.3. *Let* $r \in \mathbb{Z}_{\geq 1}$. *Let* $\boldsymbol{\pi}_j \in \mathcal{P}_{\mathbb{Z}}^+$, *for all* $1 \leq j \leq r$, *and set* $\boldsymbol{\pi} = \prod_{j=1}^{r} \boldsymbol{\pi}_j$. *Assume also that there exists a map of* $U_q(\widetilde{\mathfrak{g}})$*–modules*

$$L_q(\boldsymbol{\pi}) \to L_q(\boldsymbol{\pi}_1) \otimes \cdots \otimes L_q(\boldsymbol{\pi}_r).$$

Then there exists a map of $\mathfrak{g}[t]$*–modules*

$$L(\boldsymbol{\pi}) \to L(\boldsymbol{\pi}_1) \otimes \cdots \otimes L(\boldsymbol{\pi}_r),$$

mapping $v_{\overline{\boldsymbol{\pi}}} \to v_{\overline{\boldsymbol{\pi}_1}} \otimes \cdots \otimes v_{\overline{\boldsymbol{\pi}_r}}$. □

3. Main theorem and proof

For the remainder of the paper let $\mathfrak{g}$ be of type A_n. Let $i \in I$ and $m \in \mathbb{Z}_{\geq 1}$. Given $\xi = (\xi_1 \geq \xi_2 \geq \ldots \geq \xi_\ell)$ a partition of m, define

$$\boldsymbol{\pi}_{i,\xi} = \prod_{j=1}^{\ell} \boldsymbol{\omega}_{i,q^{\xi_j - 1},\xi_j} \in \mathcal{P}_{\mathbb{Z}}^{+}.$$

One easily checks that the above presentation of $\boldsymbol{\pi}_{i,\xi}$ is its q-factorization. In particular, by Theorem 2.1, we have

$$L_q(\boldsymbol{\pi}_{i,\xi}) \cong_{U_q(\tilde{\mathfrak{g}})} L_q(\boldsymbol{\omega}_{i,q^{\xi_1-1},\xi_1}) \otimes \cdots \otimes L_q(\boldsymbol{\omega}_{i,q^{\xi_\ell-1},\xi_\ell}) \cong_{U_q(\mathfrak{g})} V_q(\xi_1\omega_i) \otimes \cdots \otimes V_q(\xi_\ell\omega_i). \quad (3.1)$$

We state the main result of the paper.

Theorem 3.1. *Let $i \in I$, $m \in \mathbb{Z}_{\geq 1}$ and $\xi = (\xi_1 \geq \cdots \geq \xi_\ell)$ be a partition of m. Then*

$$L(\boldsymbol{\pi}_{i,\xi}) \cong_{\mathfrak{g}[t]} V(\xi_1\omega_i) * \cdots * V(\xi_\ell\omega_i).$$

Remark 3.2. By the definition of Kirillov-Reshetikhin modules and Theorem 2.1(i), if

$$V = L_q(\boldsymbol{\omega}_{i,a_1,r_1}) \otimes \cdots \otimes L_q(\boldsymbol{\omega}_{i,a_\ell,r_\ell}),$$

for some $(a_j, r_j) \in \mathbb{C}(q)^\times \times \mathbb{Z}_{\geq 1}$, then

$$V \cong_{U_q(\mathfrak{g})} V_q(r_1\omega_i) \otimes \cdots \otimes V_q(r_\ell\omega_i).$$

Therefore, setting ξ to be the partition of $r_1 + \cdots + r_\ell$ whose parts are r_j, $1 \leq j \leq \ell$, (3.1) implies that $L_q(\boldsymbol{\pi}_{i,\xi})$ is a representative of the equivalence class of V which admits graded limit.

The following is a straightforward consequence of Theorem 3.1 and [32, Theorem 3.1].

Corollary 3.3. *Let $\boldsymbol{\pi} \in \mathcal{P}_{\mathbb{Z}}^{+}$ satisfying the hypothesis of Theorem 3.1 and set $L_j = \xi_j + \cdots + \xi_\ell$, $1 \leq j \leq \ell$. Then $L(\boldsymbol{\pi})$ is isomorphic to the $\mathfrak{g}[t]$-module generated by a vector v with relations*

$$\begin{gathered}
\mathfrak{n}^+[t]v = 0, \quad (h \otimes t^s)v = \delta_{s0} L_1 \omega_i(h) v, \quad \text{for} \quad h \in \mathfrak{h},\ s \in \mathbb{Z}_{\geq 0} \\
x_\alpha^- \otimes \mathbb{C}[t]v = 0 \ \text{ for } \alpha \in R^+ \text{ with } \omega_i(h_\alpha) = 0, \\
(x_\alpha^-)^{L_1+1} v = 0 \ \text{ for } \alpha \in R^+ \text{ with } \omega_i(h_\alpha) = 1, \\
(x_\alpha^+ \otimes t)^s (x_\alpha^-)^{r+s} v = 0 \ \text{ for } \alpha \in R^+,\ r, s \in \mathbb{Z}_{\geq 1},\ \text{with } \omega_i(h_\alpha) = 1,
\end{gathered}$$

such that $r + s \geq 1 + kr + L_{k+1}$, for some $k \in \mathbb{Z}_{\geq 1}$. □

Before proving Theorem 3.1 we need to set up some notation. We shall also write ξ as the sequence $m_1^{b_1} m_2^{b_2} \ldots m_s^{b_s}$, such that $m_j \in \{\xi_1, \ldots, \xi_\ell\}$, $1 \leq j \leq s$, $0 < m_1 < m_2 < \ldots < m_s$, and $b_j > 0$ is the number of times that the integer m_j occurs in ξ.

Associated to the pair (i, ξ) we can also consider a generalized Demazure module, which we denote by $D_i(\xi)$, defined by

$$D_i(\xi) = D(t_{-m_1\omega_{i^*}}(b_1\Lambda_0), t_{-m_2\omega_{i^*}}(b_2\Lambda_0), \ldots, t_{-m_s\omega_{i^*}}(b_s\Lambda_0)),$$

where $i^* = n+1-i$, for all $i \in I$. We recall that $w_0\omega_i = -\omega_{i^*}$, for all $i \in I$. Using (1.1), we have

$$t_{-m_j\omega_{i^*}}(b_j\Lambda_0) \equiv -m_jb_j\omega_{i^*} + b_j\Lambda_0 \mod \mathbb{C}\delta, \quad j = 1, \dots, s,$$

and then, by (1.2), we have

$$D_i(\xi) = U(\mathfrak{g}[t])(v_1 \otimes \cdots \otimes v_s) \subseteq D(b_1, b_1m_1\omega_i) \otimes \cdots \otimes D(b_s, b_sm_s\omega_i),$$

where we write for short v_j instead $v_{b_j, b_jm_j\omega_i}$, for all $1 \le j \le s$.

Let ξ' denote the conjugate partition of ξ, i.e., $\xi' = n_1^{\ell_1}n_2^{\ell_2}\dots n_s^{\ell_s}$ is such that

$$n_j = \sum_{k=s-j+1}^{s} b_k \quad \text{and} \quad \ell_j = m_{s-j+1} - m_{s-j}, \quad \text{for all} \;\; j = 1, \dots, s, \tag{3.2}$$

where $m_0 = 0$.

Since $n_1 < \cdots < n_s$, Lemma 1.1 implies that $\ell(t_{-n_j\omega_{i^*}}) = \sum_{k=1}^{j} \ell(t_{-(n_k - n_{k-1})\omega_{i^*}})$, for all $1 \le j \le s$, where $n_0 = 0$. Therefore, the following theorem is straightforward from [30, Proposition 2.7], [32, Theorem 2.1] and [32, Remark 3.2].

Theorem 3.4. *Let $i \in I$, $m \in \mathbb{Z}_{\ge 0}$ and $\xi = (\xi_1 \ge \cdots \ge \xi_\ell)$ be a partition of m. Then*

$$D_i(\xi') \cong_{\mathfrak{g}[t]} V(\xi_1\omega_i) * \cdots * V(\xi_\ell\omega_i).$$

□

Note that

$$\dim(L(\boldsymbol{\pi}_{i,\xi})) = \dim(V(\xi_1) * \cdots * V(\xi_\ell)),$$

by (1.6), (3.1) and Theorem 2.2(ii). Therefore, using Theorem 3.4, to prove Theorem 3.1 it suffices to prove the following:

Proposition 3.5. *Let $i \in I$, $m \in \mathbb{Z}_{\ge 1}$ and ξ be a partition of m. There exists a surjective $\mathfrak{g}[t]$-module homomorphism*

$$L(\boldsymbol{\pi}_{i,\xi}) \twoheadrightarrow D_i(\xi').$$

We devote the remainder of this section to prove Proposition 3.5. Write $\xi = m_1^{b_1}m_2^{b_2}\dots m_s^{b_s}$, $s \in \mathbb{Z}_{\ge 1}$, and its dual $\xi' = n_1^{\ell_1}n_2^{\ell_2}\dots n_s^{\ell_s}$. Set

$$\boldsymbol{\pi}_j = \boldsymbol{\omega}_{i,a_j,\ell_j}^{n_j}, \quad \text{where} \quad a_j = q^{\ell_j - 1 + 2\sum_{k>j}\ell_k}, \; j = 1, \dots, s, \tag{3.3}$$

and observe that

$$\boldsymbol{\pi}_{i,\xi} = \prod_{j=1}^{s} \boldsymbol{\pi}_j.$$

Proposition 3.6. *Let $\boldsymbol{\pi}_j$, $1 \le j \le s$, as in (3.3). Then $L_q(\boldsymbol{\pi}_{i,\xi})$ is the unique irreducible submodule of $L_q(\boldsymbol{\pi}_s) \otimes \cdots \otimes L_q(\boldsymbol{\pi}_2) \otimes L_q(\boldsymbol{\pi}_1)$.*

Proof. By Theorem 2.1(ii), it suffices to show that

$$\ell_k - 1 + 2\sum_{t>k}\ell_t - (\ell_j - 1 + 2\sum_{t>j}\ell_t) \ne \ell_j + \ell_k + 2 - 2p + 2g - 2i, \tag{3.4}$$

for all $1 \le j < k \le s$, $1 \le p \le \min\{\ell_j, \ell_k\}$, $i < g+1 \le \min\{2i, n+1\}$. This is clear, since the left hand side of (3.4) is a negative integer and the right hand side of (3.4) is always a non-negative integer. □

Proof of Proposition 3.5. By Proposition 3.6 and Lemma 2.3, there exists a map

$$L(\boldsymbol{\pi}_{i,\xi}) \to \bigotimes_{j=1}^{s} L(\boldsymbol{\pi}_j), \tag{3.5}$$

mapping $v_{\overline{\boldsymbol{\pi}}_{i,\xi}}$ to $v_{\overline{\boldsymbol{\pi}}_s} \otimes \cdots \otimes v_{\overline{\boldsymbol{\pi}}_1}$. We claim that

$$L(\boldsymbol{\pi}_j) \cong_{\mathfrak{g}[t]} D(\ell_j, \ell_j n_j \omega_i), \quad \text{for all } 1 \le j \le s.$$

Assuming the claim, by (3.5), we have a $\mathfrak{g}[t]$-module homomorphism

$$L(\boldsymbol{\pi}_{i,\xi}) \to D(\ell_1, \ell_1 n_1 \omega_i) \otimes \cdots \otimes D(\ell_s, \ell_s n_s \omega_i),$$

whose image is $D_i(\xi')$, as required.

For the claim, let $1 \le j \le s$ and set

$$W = L_q(\boldsymbol{\varpi}_{\ell_j - 1}) \otimes \cdots \otimes L_q(\boldsymbol{\varpi}_0), \quad \text{where} \quad \boldsymbol{\varpi}_k = (\boldsymbol{\omega}_{i, a_j q^{\ell_j - 1 - 2k}})^{n_j}, \quad 0 \le k \le \ell_j - 1.$$

Observe that $\boldsymbol{\pi}_j = \prod_{k=0}^{\ell_j - 1} \boldsymbol{\varpi}_k$ and, by Theorem 2.1(ii),

$$L_q(\boldsymbol{\varpi}_k) \cong L_q(\boldsymbol{\omega}_{i, a_j q^{\ell_j - 1 - 2k}})^{\otimes n_j}, \quad \text{for all } \ 0 \le j \le \ell_j - 1.$$

Therefore, by Theorem 2.2 and Proposition 1.3, we have

$$L(\boldsymbol{\varpi}_k) \cong_{\mathfrak{g}[t]} D(1, n_j \omega_i), \quad \text{for all } 0 \le k \le \ell_j - 1.$$

Moreover, arguing as in the proof of Proposition 3.6, we obtain that $L_q(\boldsymbol{\pi}_j)$ is the unique irreducible submodule of W and, hence, there exists a $\mathfrak{g}[t]$-module homomorphism

$$L(\boldsymbol{\pi}_j) \to D(1, n_j \omega_i)^{\otimes \ell_j},$$

whose image is $D(\ell_j, \ell_j n_j \omega_i)$, by Lemma 1.4. To conclude that such surjective homomorphism is also injective, it suffices to show that

$$\dim L(\boldsymbol{\pi}_j) = \dim D(\ell_j, \ell_j n_j \omega_i). \tag{3.6}$$

Setting the partition $\psi = \ell_j^{n_j}$, we have $\psi' = n_j^{\ell_j}$ and, by Theorem 3.4, it follows that

$$D_i(\psi') = D(\ell_j, \ell_j n_j \omega_i) \cong V(\ell_j \omega_i) * \cdots * V(\ell_j \omega_i).$$

On the other hand, by Theorem 2.1,

$$L_q(\boldsymbol{\pi}_j) \cong_{U_q(\widetilde{\mathfrak{g}})} L_q(\boldsymbol{\omega}_{i, a_j, \ell_j})^{\otimes n_j} \cong_{U_q(\mathfrak{g})} V_q(\ell_j \omega_i)^{\otimes n_j}.$$

By Theorem 2.2(i) and (ii), and using (1.6) we conclude that (3.6) holds, which finishes the proof. □

Remark 3.7. Assume $\mathfrak{g} = \mathfrak{sl}_2$ and let $I = \{1\}$. It is well known that if $\boldsymbol{\pi} = \prod_{j=1}^{m} \boldsymbol{\omega}_{1, a_j, r_j}$ is the q-factorization of $\boldsymbol{\pi}$, then

$$L_q(\boldsymbol{\pi}) \cong_{U_q(\widetilde{\mathfrak{g}})} \otimes_{j=1}^{m} L_q(\boldsymbol{\omega}_{1, a_j, r_j}).$$

In particular, given $m \in \mathbb{Z}_{\ge 0}$, each affinization of $V_q(m\omega_1)$ must be of this form (see [5, Lemma 6.5]). Therefore, the equivalence classes of affinizations of $V_q(m\omega_1)$ are in bijection with the

set of all partitions ξ of m, and each of these classes has $L_q(\boldsymbol{\pi}_{1,\xi})$ as its representative. In particular, Theorem 3.1 implies that the graded limit of each class of affinization of $V_q(m\omega_1)$ is isomorphic to a fusion product.

It is also known that this statement does not hold in general. For instance, if $\mathfrak{g} = \mathfrak{sl}_4$ it can easily be proved that we have three different classes of affinizations of $V_q(2\omega_2)$, with representatives $L_q(\boldsymbol{\omega}_{2,q,2})$, $L_q(\boldsymbol{\omega}_{2,1}\boldsymbol{\omega}_{2,q^4})$ and $L_q(\boldsymbol{\omega}_{2,1}\boldsymbol{\omega}_{2,q^6})$, for example. By Theorems 3.1 and 3.4, we have

$$L(\boldsymbol{\omega}_{2,q,2}) \cong_{\mathfrak{g}[t]} V(2\omega_2) \cong_{\mathfrak{g}[t]} D(2,2\omega_1) \quad \text{and} \quad L(\boldsymbol{\omega}_{2,1}\boldsymbol{\omega}_{2,q^6}) \cong_{\mathfrak{g}[t]} V(\omega_2) * V(\omega_2) \cong_{\mathfrak{g}[t]} D(1,2\omega_2).$$

Using the results of [33] we have that $L_q(\boldsymbol{\omega}_{2,1}\boldsymbol{\omega}_{2,q^4})$ is a module corresponding to a skew Young diagram and

$$L_q(\boldsymbol{\omega}_{2,1}\boldsymbol{\omega}_{2,q^4}) \cong_{U_q(\mathfrak{g})} V_q(2\omega_2) \oplus V_q(\omega_1+\omega_3).$$

Moreover, since $D(2,2\omega_2) \cong_{\mathfrak{g}[t]} U(\mathfrak{g}[t])(v_{1,\omega_2} \otimes v_{1,\omega_2}) \subseteq D(1,\omega_2) \otimes D(1,\omega_2)$, by Lemma 1.4, we cannot have $L(\boldsymbol{\omega}_{2,1}\boldsymbol{\omega}_{2,q^4})$ being isomorphic to a fusion product and nor a generalized Demazure module, but rather a proper quotient of $D(1,2\omega_2)$.

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